Triple integrals: Same as before, but with 3 variables

$\iiint_R f(x,y,z)\,dV$ is like the height over a rectangular box

ex: $\iiint_E (xy+z^2)\,dV$ $E=\underset{x}{[0,2]}\times\underset{y}{[0,1]}\times\underset{z}{[0,3]}$ region is a rectangular prism

$\int_0^2\int_0^1\int_0^3 (xy+z^2)\,dz\,dy\,dx = \int_0^2\int_0^1 \left[xyz+\frac{z^3}{3}\right]_0^3 dy\,dx$

$= \int_0^2\int_0^1 3xy+9\,dy\,dx = \int_0^2 \left[\frac{3}{2}xy^2+9y\right]_0^1 dx = \int_0^2 \frac{3}{2}x+9\,dx$

$= \left[\frac{3}{4}x^2+9x\right]_0^2 = 3+18 = 21$

$\iiint_R (2x-y)\,dV$ $R=\{(x,y,z): \underset{①}{0\le z\le 2},\ \underset{②}{0\le y\le z^2},\ \underset{③}{0\le x\le y-z}$

which is in the form: $c_1\le z\le c_2$, $g(z)_1\le y\le g(z)_2$, $h(y,z)_1\le x\le h_2(y,z)$

① is constant, ② is a function of z ③ is a function of y, z

easy to integrate in order ③, ②, ① because is in terms of successively lower amounts of variables

$\int_0^2\int_0^{z^2}\int_0^{y-z} (2x-y)\,dx\,dy\,dz = \int_0^2\int_0^{z^2}\left[x^2-xy\right]_0^{y-z}$

$= \int_0^2\int_0^{z^2} (z^2-yz)\,dy\,dz = \int_0^2 \left[yz^2-\frac{zy^2}{2}\right]_0^{z^2} = \int_0^2 z^4-\frac{1}{2}z^5\,dz$

$= \left[\frac{z^5}{5}-\frac{z^6}{12}\right]_0^2 = \frac{16}{15}$

what if the order was switched?

if: $\int_0^2\int_0^{y-z}\int_0^{z^2} (2x-y)\,dy\,dx\,dz = \int_0^2\int_0^{y-z}\left[2xy-\frac{y^2}{2}\right]_0^{z^2}$

$= \int_0^2\int_0^{y-z} 2xz^2-\frac{z^4}{2}\,dx\,dz = \int_0^2 \left[x^2z^2-\frac{xz^4}{2}\right]_0^{y-z}$

$= \int_0^2 \underset{\uparrow}{(y-z)^2}z^2-\frac{1}{2}\underset{\uparrow}{(y-z)}z^4\,dz$ reparametrized to make y bounded by x

these can't go away unless

if bounding y by x: $0 \leq z \leq 2$, $0 < y \leq z^2$, $0 \leq x < y-z$

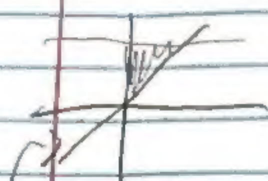
endpoints: $x=0$ $\quad x=y-z \Rightarrow x=z^2-z \Rightarrow 0 \leq x \leq z^2-z$

$y=0 \quad y=z^2 \qquad\qquad x+z \leq y \leq z^2$

$z=0 \quad z=2 \qquad R\ \{(x,y,z) : 0 \leq z \leq 2,$

$0 \leq x \leq z^2-z$

$0 < y \leq z$



$$\iiint_R (2x-y)\,dV = \int_0^2 \int_0^{z^2-z} \int_{x+z}^{z^2} (2x-y)\,dy\,dx\,dz$$

$$= \int_0^2 \int_0^{z^2-z} \left[2xy - \frac{y^2}{2}\right]_{x+z}^{z^2} dx\,dz = \int_0^2 \int_0^{z^2-z} \left(2xz^2 - \frac{z^4}{2}\right) - \left(2x(x+z) - \frac{(x+z)^2}{2}\right) dx\,dz$$

$$= \int_0^2 \int_0^{z^2-z} \left(\frac{-3x^2}{2} + \frac{z^2}{2} + 2xz^2 - \frac{z^4}{2} - xz\right) dx\,dz = \int_0^2 \left[x^2z^2 - \frac{xz^4}{2} - \frac{x^3}{2} - \frac{x^2z}{2} + \frac{xz^2}{2}\right]_0^{z^2-z}$$

$$= \int_0^2 (z^2-z)(z^2-z)z - \frac{1}{2}z^4 - \frac{1}{2}(z^2-z)^2 - \frac{1}{2}z + \frac{1}{2}z^2$$ *many mistakes later*

reparameterization is tough, see website

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compute volume of tetrahedron with vertices: $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

$\mathrm{vol}(T) = \iiint_T 1\,dV \qquad x+y+z=1$

$0 \leq x \leq 1$, $0 \leq y \leq 1-x$, $0 \leq z \leq 1-x-y$

$$\text{So } \mathrm{vol}(T) = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\,dz\,dy\,dx = \int_0^1 \int_0^{1-y} [z]_0^{1-x-y}\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx = \int_0^1 \left[y - xy - \frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1 \left(1-x - x(1-x) - \frac{1}{2}(1-x)^2\right) dx = \frac{1}{2}\int_0^1 (1-x)^2\,dx$$

$$= -\frac{1}{2}\cdot\frac{1}{3}\left[(1-x)^3\right]_0^1 = -\frac{1}{6}(-1) = \frac{1}{6}$$